QP CODE: 26022401

26022401

Reg No : .....

Name : .....

# M Sc DEGREE (CSS) EXAMINATION MAY 2026

## Second Semester

M Sc MATHEMATICS

## CORE COURSE - ME010205 - MEASURE AND INTEGRATION

2019 ADMISSION ONWARDS

7B6635E9

Time: 3 Hours

Weightage: 30

### Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Show that every interval is a Borel set.
2. 1. When will you say a property holds almost everyehere on a measurable set $E$?
   2. Let $\{E_k\}_{k=1}^{\infty}$ is a countable collection of Lebesgue measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then prove that almost all $x \in \mathbb{R}$ belong to at most finitely many of the $E_k$'s.
3. Prove that there is a measurable set which is not a Borel set
4. Define Riemann Integrablity of $f$ over $[a, b]$. Show that there exist functions which are not Riemann integrable.
5. Show with an example that Lebesgue integrablity of bounded function $f$ defined on a closed bounded interval $[a, b]$ need not imply Riemann integrablity of $f$.
6. Define integrablity of a Lebesgue measurable function. State the general Lebesgue dominated convergence theorem for a sequence of Lebesgue measurable functions on E.
7. State and prove the Borel-Cantelli lemma for general measures.
8. Define general outer measure and measurability.
9. State and prove Chebychev's Inequality.
10. Give an example of two mutually singular measures in a measure space.

(8×1=8 weightage)

## Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Prove that the Lebesgue outer measure of $[a, b]$ is $b - a$

12. Let $E$ be any set of real numbers. Then prove that $E$ is Lebesgue measurable if and only if for each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing $E$ for which $m^*(\mathcal{O} - E) < \epsilon$

13. Prove that the linear combination of two Lebesgue measurable functions $f$ and $g$ that are finite a.e. on $E$, is measurable on $E$ . Also prove that the product $fg$ is Lebesgue measurable.

14. Prove that an extended real valued function $f$ on a Lebesgue measurable set $E$ is Lebesgue measurable if and only if there is a sequence of simple functions $\{\phi_n\}$ on $E$, which converges pointwise on $E$ to $f$ and has the property that $|\phi_n| \leq |f|$ on $E$ for all $n$.

15. Prove that every measurable subset of a negative set is negative and the countable union of negative sets is negative.

16. Let $(X, \mathcal{M}, \mu)$ be a general measure space and $\{f_n\}$ be a sequence of measurable functions on $X$ for which $f_n \to f$ pointwise a.e. on $X$. Prove that, if either $(X, \mathcal{M}, \mu)$ is a complete measure space or the convergence is pointwise on all of $X$, then $f$ is measurable.

17. Let $(X, \mathcal{M}, \mu)$ be a measure space and the function $f$ be integrable over $X$. Then prove the following statements.

    1. If $\{X_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of $X$ whose union is $X$, then $\int_X f \, d\mu = \lim_{n\to\infty} \int_{X_n} f \, d\mu$
    2. If $\{X_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of $X$ and $B = \bigcap_{n=1}^{\infty} X_n$, then $\int_B f \, d\mu = \lim_{n\to\infty} \int_{X_n} f \, d\mu$

18. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of functions on $X$ that is both uniformly integrable and tight over $X$. Assume $\{f_n\} \to f$ pointwise a.e. on $X$ and the function $f$ is integrable over $X$. Then prove that $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$

(6×2=12 weightage)

## Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19.

    1. Define algebra and $\sigma$-algebra.
    2. Prove that the collection $\mathcal{M}$ of measurable sets is a $\sigma$-algebra that contains the $\sigma$-algebra $\mathcal{B}$ of Borel sets.
    3. Prove that every $G_\delta$ sets and $F_\sigma$ sets are measurable.

20. Prove that Lebesgue integration of nonnegative Lebesgue measurable functions on E is
1) Linear 2) Monotonic 3) Additive over domains of integration.

21. (i) State and prove the Hahn decomposition theorem.
(ii) Prove that the Hahn decomposition is unique except for null sets.

22. (a) State Radon Nikodym Theorem
(b) Write a short note on Radon Nikodym Derivative.
(c) State Lebesgue Decomposition theorem

(2×5=10 weightage)